

MT Lecture 2 - Induction

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Consider the following statement $P(n)$:

$$P(n) : \quad n^2 < 2^n \text{ for all } n > 4, n \in \mathbb{N}$$

In proving this statement, one could convert the statement to its stronger continuous analogue,

$$x^2 < 2^x \text{ for all } x > 4, x \in \mathbb{R}$$

and use calculus, then restrict back to the natural numbers. In fact, we will do that right now. Take 5 min.

Back to the original statement. Rather than using heavy calculus machinery, we use induction. First, show the statement holds for $n = 5$ (base case). Then show that if the statement is true for n , it must be true for $n + 1$ (inductive step). In this way we create a logical chain which extends to infinity:

$$(\text{base case}) P(5) \equiv \text{True} \longrightarrow P(6) \equiv \text{True} \longrightarrow P(7) = \text{True} \rightarrow \dots$$

Assuming that the statement is true **for some value of n** is called the Inductive Hypothesis (IH). The choice of this n is always general and never specified. Showing that this assumption implies the truth of the statement for $n+1$ is the key step of induction. The base case then actually specifies a specific value of n and shows the truth for this "case", which starts the truth chain whose engine is powered by the inductive step. Lets write a proof sketch.

proof sketch. The statement holds for $n = 5$, as $5^2 = 25 < 32 = 2^5$ (base case). Now suppose that the statement holds for some arbitrary n , ie

$$n^2 < 2^n \text{ for some } n > 4, n \in \mathbb{N}$$

We want to show that this assumption implies that

$$(n+1)^2 < 2^{n+1}$$

Lets unpack the LHS (left hand side):

$$(n+1)^2 = n^2 + 2n + 1 <^{(IH)} 2^n + 2n + 1$$

Lets unpack the RHS:

$$2^{n+1} = 2 \cdot 2^n = 2^n + 2^n$$

Hence we need to show $2^n > 2n + 1$. We could use another induction, but this would waste time. Notice that we still have the IH at our disposal, so we know that $2^n > n^2$. Thus we just need $n^2 > 2n + 1$ for $n > 4$. This is a polynomial in n , so now(!) we can convert to continuous analogue and argue that

$$x^2 - 2x - 1 > (x-4)(x+2) > 0 \quad \text{for all } x > 4, x \in \mathbb{R}$$

$$\implies n^2 > 2n + 1 \quad \text{for all } n > 4, n \in \mathbb{N}$$

This completes all the pieces we need to write a formal proof. When writing the proof, specify that you are using induction to make the steps clearer to the reader.

Proof by Induction.

The statement holds for $n = 5$, as $5^2 = 25 < 32 = 2^5$ (base case).

Now suppose that the statement holds for some $n \in \mathbb{N}$,

$$n^2 < 2^n \text{ for some } n > 4, n \in \mathbb{N} \quad (\text{IH})$$

Then we have

$$(n+1)^2 = n^2 + 2n + 1 < n^2 + n^2 < 2^n + 2^n = 2^{n+1}$$

Where the second inequality follows from (IH), and the first inequality holds since $n^2 > 2n + 1$, as

$$n^2 - 2n - 1 > (n-4)(n+2) > 0 \quad \text{for all } n > 4$$

Thus, by induction, we have shown the truth of the statement:

$$n^2 < 2^n \text{ for all } n > 4, n \in \mathbb{N}$$

□

Now you try. Prove similar statements for $n^3 < 3^n$ (find the appropriate lower bound on n). Prove directly for $n^{2024} < 2024^n$. Can you do the general case $n^m < m^n$ for some fixed m ? What are the restrictions we need to place on n, m so that this general statement holds? Write the proofs using calculus and/or induction. Take 20 min.

This was an example of when induction is NOT the right proof method. Observe how quickly one can use calculus to solve:

Proof. The function $\frac{x}{\log(x)}$ is increasing (and continuous) for $x \geq 3$ since

$$\left(\frac{x}{\log(x)} \right)' = \frac{\log(x) - 1}{\log(x)^2} > 0 \quad \forall x > e$$

Thus $m < n$ implies that

$$\frac{m}{\log(m)} < \frac{n}{\log(n)}$$

Hence

$$\log(n^m) < \log(m^n)$$

Since $\log(x)$ is continuous and monotone increasing for $x > 0$, we conclude that

$$n^m < m^n \quad \forall n > m \geq 3$$

□

Strong Induction

Sometimes the inductive hypothesis is not strong enough to write a proof. There is a stronger inductive hypothesis that you can make, which is completely equivalent to regular induction, but may afford more flexibility.

STRONG IH (SIH): Suppose the statement holds **for some** $N \in \mathbb{N}$, and all $n \leq N$.

The only difference is you add the assumption of $n \leq N$ also being true, which is allowed since the "truth chain engine", $n \equiv \text{true} \rightarrow n + 1 \equiv \text{true}$, powered by the inductive step, moves only forward and technically assumes everything behind it to be true (so we may as well use this assumption). Why is this useful? Do the following problems.

Show that

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$$

Prove that every positive integer n can be written as a sum of distinct nonnegative integer powers of 2.

Fundamental Theorem of Arithmetic. Every positive integer can be written as a product of prime factors, and this product is unique up to reordering of the factors.

<https://brilliant.org/wiki/strong-induction/>

Backward Induction

Sometimes even strong induction is not enough. There are some tricky methods described in Putnam and Beyond using backwards induction, which is the following:

1. Show that there is a sequence $\{n_k\} \rightarrow \infty$ such that the statement is true for all n_k .

(usually using induction on k , not n !)

2. Show that $n \rightarrow n - 1$.

Common examples of sequences to use are the even integers, or powers of 2, but the choice is flexible and usually depends on the problem. For example, if I show that the statement holds for all $n = 2^k$, $k \in \mathbb{N}$ and complete a backwards induction step $n \rightarrow n - 1$, the statement must be true for all n .

See Putnam and Beyond 31,32