MT Lecture 1 - Contradiction

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Proofs by contradiction naturally arise when a statement is impossible to prove by brute force. Here is the structure:

1. Start by assuming the statement is false.

2. Show that this assumption produces a logical contradiction,
i.e. forces something you know is true to be false.

3. Since a contradiction was produced, we conclude
that the initial assumption 1. must be wrong,
i.e. that the statement is in fact true.

Example. Show that there are infinitely many prime numbers.

Example. Show that $\sqrt{2}$ is irrational.

Example. Show that $\sqrt{3}$ is irrational.

Proposition. For all real numbers x and y, if $x \neq y$, $x > 0$, and $y > 0$, then $\frac{x}{y} + \frac{y}{x} > 2$.

proof. Suppose for sake of contradiction that

$$\frac{x}{y} + \frac{y}{x} \leq 2$$

$$\frac{x^2 + y^2}{xy} \leq 2$$

$$x^2 + y^2 \leq 2xy$$

$$x^2 - 2xy + y^2 \leq 0$$

$$(x - y)^2 \leq 0$$

Which is a contradiction since $x \neq y$.

Why did we need the contradiction structure here? Answer is we didn't. However, supposing the contradiction *led us to the right algebraic ideas.* So contradiction proofs can always be useful to consider, even if they are somewhat unnecessary.

The nice part about proofs by contradiction is we are supposing the statement to be false initially, which is a definitive claim. In a direct proof, you must NEVER begin by assuming what you want to show is true. That is, one should never make premature definitive claims about the statement unless it is a proof by contradiction!

Consider the alternate INCORRECT proof:

$$\frac{x}{y} + \frac{y}{x} > 2$$

$$\frac{x^2 + y^2}{xy} > 2$$

$$x^2 + y^2 > 2xy$$

$$x^2 - 2xy + y^2 > 0$$

$$(x - y)^2 > 0$$

Which is true since $x \neq y$.

Explain why this proof is wrong and fix it. Observe how we needed the logic of this proof to be **reversible**, which means that every step of the incorrect proof is actually an if and only if (iff) statement, so the steps of the incorrect proof can be written in reverse order and the proof becomes correct. However, if the logic is not reversible, then this would fail which means it is extremely dangerous and very wrong to start a proof by assuming the statement is true.

Now lets do some much, much harder problems.

Putnam and Beyond 3,4,5,6.

Spend the rest of the time working on Putnam and Beyond 1-25, which will spill over into induction (which will be covered on Wed).